International Journal of Theoretical Physics, Vol. 36, No. 3, 1997

Multiple Lax Pair for the Double Ernst Equation

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Received June 24, 1996

The double-complex function method is used to find a multiple Lax pair for the double Ernst equation. Then multiple hidden symmetries and multiple infinite sequences of nonlocal conserved charges are obtained. The result of Papachristou and Harrison is contained in our result.

Papachristou and Harrison (1994) proposed a Lax pair which unifies some symmetries and integrability properties of the stationary axisymmetric vacuum gravitational field (SAVGF) equations. By using this Lax pair some new hidden symmetries and infinite sequences of nonlocal conserved charges can be constructed. However, in this scheme only ordinary (complex) numbers were used. According to the double-complex function method (Zhong, 1985, 1990), if the double symmetry of the SAVGF equations is fully used, additional hidden symmetries may be found. In the present article, we use the double-complex function method to extend the scheme of Papachristou and Harrison (1994) into a multiple form and obtain multiple hidden symmetries as well as multiple infinite sequences of nonlocal conserved charges for the SAVGF; the result of Papachristou and Harrison (1994) is contained in ours.

First we review some relevant notations and results of the double-complex method (Zhong, 1985). Let J denote the double imaginary unit, i.e., $J = i$ ($i^2 = -1$) or $J = \epsilon$ ($\epsilon^2 = +1$, $\epsilon \neq \pm 1$). When a real series $\Sigma\, a_n$ is absolutely convergent, $a(J) = \sum_{n=0}^{\infty} a_n J^{2n}$ is called a double-real number and we write $a_C = a(J = i)$, $a_H = a(J = \epsilon)$. If the SAVGF line element is written as

$$ds^2 = f(dt - \omega\, d\phi)^2 - f^{-1}[e^{2\gamma}(dz^2 + d\rho^2) + \rho^2\, d\phi] \tag{1}$$

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0020-7748/97/0300-0689$12.50/0

then the SAVGF equations can be reduced to the double matrix form (Zhong, 1988; Gao and Zhong, 1992, 1996)

$$\partial_\rho[\rho M^{-1}(J)\partial_\rho M(J)] + \partial_z[\rho M^{-1}(J)\partial_z M(J)] = 0 \tag{2a}$$

$$M^*(J) = M(J), \qquad M^T(J) = M(J), \qquad \det M(J) = -J^2 \tag{2b}$$

where $M(J)$ is a 2×2 double matrix function of ρ and z. Equation (2b) implies that $M(J)$ can be written as

$$M(J) = \frac{1}{F(J)}\begin{pmatrix} 1 & \Omega(J) \\ \Omega(J) & \Omega^2 - J^2F(J) \end{pmatrix} \tag{3}$$

in terms of two double-real functions $F(J)$ and $\Omega(J)$. If a double solution $M(J)$ of equation (2) is given, then, we can obtain via equation (3) a pair of dual SAVGF solutions for equation (1) as follows:

$$(f, \omega) = (F_C, V_{F_C}(\Omega_C)), \qquad (\hat{f}, \hat{\omega}) = (T(F_H), \Omega_H) \tag{4}$$

where the NK transformations T, V are defined as

$$T: \quad F \to T(F) = \rho/F$$

$$V: \quad F, \Omega \to V_F(\Omega) = \int \frac{\rho}{F^2}(\partial_z\Omega\, d\rho - \partial_\rho\Omega\, dz) \tag{5}$$

Let $\delta M(J) = \alpha Q(J)$ denote an infinitesimal symmetry transformation of equations (2), where α is an infinitesimal real parameter and $Q(J)$ a double matrix function of ρ and z. Similar to Papachristou and Harrison (1994), if we set $\Phi(J) = M^{-1}(J)Q(J)$ and define two double linear operators $D_\rho(J)$ and $D_z(J)$ by

$$D_\rho(J) = \rho(\partial_\rho + [M^{-1}(J)\partial_\rho M(J), \cdot])$$

$$D_z(J) = \rho(\partial_z + [M^{-1}(J)\partial_z M(J), \cdot]) \tag{6}$$

then the general symmetry condition for equation (2a) can be stated as

$$\partial_\rho D_\rho(J)\Phi(J) + \partial_z D_z(J)\Phi(J) = 0 \tag{7}$$

and equation (2b) requires that

$$Q^*(J) = Q(J), \qquad Q^T(J) = Q(J), \qquad \operatorname{tr}\Phi(J) = 0 \tag{8}$$

Furthermore, through direct calculations we find that the double operator identity

$$[D_\rho(J), D_z(J)] = D_z(J) \tag{9}$$

is satisfied and that equation (2a) is equivalent to the double operator equation

$$[D_\rho(J), \partial_\rho] + [D_z(J), \partial_z] + \partial_\rho = 0 \tag{10}$$

Now, similar to Zhong (1990), we introduce a so-called double ordinary-complex, single-valued 2×2 matrix function $\Psi(\lambda; J) = \Psi(\rho, z, \lambda; J)$ of ρ, z, and an ordinary-complex parameter λ, i.e., $\Psi(\lambda; J)$ has the form

$$\Psi(\lambda; J) = A(\lambda; J) + iB(\lambda; J) \tag{11}$$

where $A(\lambda; J)$, $B(\lambda; J)$ are both double-real matrix functions when λ is restricted to **R** (real numbers). Then we consider the following double linear system:

$$D_\rho(J)\Psi(\lambda; J) - 2\lambda\partial_\lambda\Psi(\lambda; J) = \frac{1}{\lambda}\,\partial_z\Psi(\lambda; J)$$

$$D_z(J)\Psi(\lambda; J) = -\frac{1}{\lambda}\,\partial_\rho\Psi(\lambda; J) \tag{12}$$

and from equations (9) and (10) we obtain the following result.

Proposition 1. The linear system (12) is integrable for $\Psi(\lambda; J)$ if $M(J)$ is a solution of equation (2a). ■

Moreover, we demand further that $\Psi(\lambda; J)$ be analytic in a deleted neighborhood D around the origin of the λ-plane. From equation (11), the meaning of the analyticity of $\Psi(\lambda; J)$ is clear. Thus we expand $\Psi(\lambda; J)$ into a Laurent series in D

$$\Psi(\lambda; J) = \sum_{n=-\infty}^{\infty} \lambda^n \psi^{(n)}(J) \tag{13}$$

$$\psi^{(n)}(J) = \psi^{(n)}(\rho, z; J) = \frac{1}{2\pi i}\oint_L \frac{d\lambda}{\lambda^{n+1}}\,\Psi(\lambda; J) \tag{14}$$

where L, lying entirely in D, is a positively oriented, simple closed contour around $\lambda = 0$. Substituting equation (13) into (12), we obtain

$$[D_\rho(J) - 2n]\psi^{(n)}(J) = \partial_z\psi^{(n+1)}(J)$$

$$D_z(J)\psi^{(n)}(J) = -\partial_\rho\psi^{(n+1)}(J), \qquad n = 0, \pm 1, \pm 2, \ldots \tag{15}$$

Equations (15) can be regarded as Backlund transformations relating $\psi^{(n)}(J)$ and $\psi^{(n+1)}(J)$ and depending parametrically on $M(J)$. From (15) we have the following result.

Proposition 2. $\psi^{(n)}(J)$ ($n = 0, \pm 1, \pm 2, \ldots$) is a double infinite sequence of conserved charges for equation (2a). Moreover, $\psi^{(0)}(J)$ satisfies the symmetry condition (7), hence

$$\delta M(J) = \alpha Q(J) = \alpha M(J)\psi^{(0)}(J) \tag{16}$$

is a symmetry of equation (2a). ∎

In order to obtain symmetries of the SAVGF equations, the infinitesimal transformation (16) has to satisfy the further condition (8). First of all, we see that the operators $D_\rho(J)$, $D_z(J)$ defined by equations (6) are double-real, so the double Lax pair (12) is compatible with the constraints

$$\operatorname{tr} \Psi(\lambda; J) = 0, \qquad \Psi(\lambda^*; J)^* = \Psi(\lambda; J) \tag{17}$$

The second constraint in (17) is equivalent to the condition $B(J) = 0$ in (11). Therefore, for our purpose we seek traceless solutions of (12) which take double-real values when λ is restricted to the real axis of the complex plane. In addition, if $M(J)$ satisfies equation (2a), so does $M^T(J)$. Consequently, if $M^T(J) = M(J)$ and $\delta M(J) = \alpha Q(J)$ is a symmetry, then so is $\delta M(J) = \alpha Q^T(J)$ as well as $\delta M(J) = \alpha(Q(J) + Q^T(J))$. Thus from equations (16) and (15) we have the following result.

Proposition 3. For a known solution $M(J)$ of equations (2a) and (2b) and a solution of (12) which satisfies constraints (17), the double transformation

$$\delta M(J) = \frac{\alpha}{2\pi i} \oint_L \frac{d\lambda}{\lambda} [M(J)\Psi(\lambda; J) + \Psi^T(\lambda; J)M(J)] \tag{18}$$

is a double-real infinitesimal symmetry transformation leaving equations (2a), (2b) invariant in form.

Proof. Let

$$Q(J) = \frac{1}{2\pi i} \oint_L \frac{d\lambda}{\lambda} [M(J)\Psi(\lambda; J) + \Psi^T(\lambda; J)M(J)]$$

Then from equations (2b) and (17) it is easily verified that the conditions $Q^T(J) = Q(J)$ and $\operatorname{tr}(M^{-1}(J)Q(J)) = 0$ are satisfied. To prove the reality of $Q(J)$, we observe that from (17), (2b), and the expression for $Q(J)$ we have

$$Q(J)^* = \frac{-1}{2\pi i} \oint_{L^*} \frac{d\lambda^*}{\lambda^*} [M(J)\Psi(\lambda^*; J) + \Psi^T(\lambda^*; J)M(J)] \tag{19}$$

where L^* is the complex conjugate contour of L. Noticing the analyticity of $\Psi(\lambda; J)$ in the annular belt D above, we can select L^* and L to be reversely oriented. Setting $\zeta = \lambda^*$, we find that (19) becomes

$$Q(J)^* = \frac{(-1)^2}{2\pi i} \oint_L \frac{d\zeta}{\zeta} [M(J)\Psi(\zeta; J) + \Psi^T(\zeta; J)M(J)]$$
$$= Q(J)$$

By this condition the corresponding transformations give us real physical solutions of the SAVGF. ■

The results above are all double; when $J = i$, they reduce to the results given by Papachristou and Harrison (1994); when $J = \epsilon$, they are new. However, we find that the SAVGF equations have more symmetries. To this end, we introduce a double dual transformation $d(J)$ as in Zhong (1990). Considering (3), we define $d(J)$ by

$$d(J)\colon\quad M(J) \to \hat{M}(J) = \frac{1}{\hat{F}(J)} \begin{pmatrix} 1 & \hat{\Omega}(J) \\ \hat{\Omega}(J) & \hat{\Omega}^2(J) - J^2\hat{F}^2(J) \end{pmatrix}$$

$$\hat{F}(J) = \rho/F(\mathring{J}), \qquad \hat{\Omega}(J) = \mathring{J}^2 \int \frac{\rho}{F^2(\mathring{J})} [\partial_z \Omega(\mathring{J})\, d\rho - \partial_\rho \Omega(\mathring{J})\, dz] \tag{20}$$

where the "∘" denotes the imaginary unit commutation operation

$$\circ\colon\quad J \mapsto \mathring{J}, \quad \mathring{i} = \epsilon, \quad \mathring{\epsilon} = i \tag{21}$$

It can be verified that if $M(J)$ is a solution of equations (2a) and (2b), so is $\hat{M}(J)$. Thus from equations (6) and (12) we have the corresponding $\hat{D}_\rho(J)$, $\hat{D}_z(J)$, and another linear system

$$\hat{D}_\rho(J)\hat{\Psi}(\lambda; J) - 2\lambda\partial_\lambda\hat{\Psi}(\lambda; J) = \frac{1}{\lambda}\partial_z\hat{\Psi}(\lambda; J)$$

$$\hat{D}_z(J)\hat{\Psi}(\lambda; J) = -\frac{1}{\lambda}\partial_\rho\hat{\Psi}(\lambda; J) \tag{22}$$

It should be pointed out that $M(J)$ and $\hat{M}(J)$ give the same SAVGF solutions, but $\hat{D}_\rho(J) \neq D_\rho(J)$ and $\hat{D}_z(J) \neq D_z(J)$, which lead to the important result that the linear systems (22) and (12) are different, i.e., $\hat{\Psi}(\lambda; J)$ and $\Psi(\lambda; J)$ satisfy different equations (Zhong, 1990) and provide different symmetries and conserved charges. Similar to the preceding discussions, we can also expand $\hat{\Psi}(\lambda; J)$ as

$$\hat{\Psi}(\lambda; J) = \sum_{n=-\infty}^{\infty} \lambda^n \hat{\psi}^{(n)}(J)$$

$$\hat{\psi}^{(n)}(J) = \frac{1}{2\pi i} \oint_L \frac{d\lambda}{\lambda^{n+1}} \hat{\Psi}(\lambda; J), \qquad n = 0, \pm 1, \pm 2, \ldots \tag{23}$$

Hence, we have the following result.

Proposition 4. $\hat{\psi}^{(n)}(J)$ $(n = 0, \pm 1, \pm 2, \ldots)$ in equation (23) is a double infinite sequence of conserved charges for the SAVGF equations and

$$\hat{\delta} M(J) \equiv \delta \hat{M}(J) = \frac{\alpha}{2\pi i} \oint_{L} \frac{d\lambda}{\lambda} [\hat{M}(J)\hat{\Psi}(\lambda; J) + \hat{\Psi}^{T}(\lambda; J)\hat{M}(J)] \quad (24)$$

is another double-real infinitesimal symmetry transformation leaving equations (2a) and (2b) invariant in form. ■

Propositions 2–4 show that, for each given double solution of equations (2a) and (2b), we can obtain four infinite sequences of conserved charges and four infinitesimal symmetry transformations simultaneously. So we say the results are multiple, a special case of which (take $J = i$ in Propositions 2 and 3) gives the result of Papachristou and Harrison (1994). The multiplicity of the present result reflects the fact that the SAVGF have more symmetries than previously expected, which was pointed out in Zhong (1990) in a different context. Taking exponential mappings of these symmetric algebras and giving some multiple groups of finite symmetric transformations will be investigated in a future paper.

Finally, for contrast and to show the use of the above multiple method, we give an example which extends directly the example of Papachristou and Harrison (1994) into a multiple form. Let $\psi^{(0)}(J) = \hat{\psi}^{(0)}(J) = g_0$ be a traceless real, constant matrix, and $M_0(J)$ a given double solution of (2) [e.g., one of the double solutions given in Zhong (1985) or Gao and Zhong (1992, 1996)]; then from equation (15) and Propositions 2 and 4 we obtain the corresponding multiple conserved charges of equation (2a):

$$\psi^{(1)}(J) = [X(J), g_0] \quad (25a)$$

$$\hat{\psi}^{(1)}(J) = [\hat{X}(J), g_0] \quad (25b)$$

$$\psi^{(2)}(J) = [Y(J), g_0] + \tfrac{1}{2}[X(J), [X(J), g_0]] \quad (26a)$$

$$\hat{\psi}^{(2)}(J) = [\hat{Y}(J), g_0] + \tfrac{1}{2}[\hat{X}(J), [\hat{X}(J), g_0]] \quad (26b)$$

etc., where $X(J)$ is the double potential of (2), defined by $\partial_\rho X(J) = -\rho M_0^{-1}(J)\partial_z M_0(J)$, $\partial_z X(J) = \rho M_0^{-1}(J)\partial_\rho M_0(J)$, and satisfying the equation

$$\rho(\partial_\rho^2 X(J) + \partial_z^2 X(J)) - \partial_\rho X(J) + [\partial_z X(J), \partial_\rho X(J)] = 0 \quad (27)$$

while $Y(J)$ is the potential of equation (27), defined by

$$\partial_\rho Y(J) = \tfrac{1}{2}[\partial_\rho X(J), X(J)] - \rho\partial_z X(J)$$

$$\partial_z Y(J) = \tfrac{1}{2}[\partial_z X(J), X(J)] + \rho\partial_\rho X(J) - 2X(J)$$

and with similar expressions for $\hat{X}(J)$ and $\hat{Y}(J)$ [cf. equation (20)]. Also, from

Propositions 3 and 4 we can obtain the corresponding multiple infinitesimal symmetric transformation of equations (2a) and (2b) as

$$\delta M(J) = \alpha[M_0(J)g_0 + g_0^T M_0(J)] \tag{28a}$$

$$\hat{\delta} M(J) = \alpha[\hat{M}_0(J)g_0 + g_0^T \hat{M}(J)] \tag{28b}$$

Upon taking $J = i$ in equations (25), (26a), and (28a) one recovers the result given in Papachristou and Harrison (1994). However, the other results above are new in spite of the fact that $\psi^{(0)}(J)$ and $\hat{\psi}^{(0)}(J)$ are taken to be the same.

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